

Lower bound of multipartite concurrence based on sub-partite quantum systems

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Abstract

We study the concurrence of arbitrary dimensional multipartite quantum systems. An explicit analytical lower bound of concurrence for four-partite mixed states is obtained in terms of the concurrences of tripartite mixed states. Detailed examples are given to show that our lower bounds improve the existing lower bounds of concurrence. The approach is generalized to arbitrary multipartite quantum systems.

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As a striking feature of quantum physics and an essential resource in quantum information processing [1]-[4], quantum entanglement has attracted much attention in recent years [5]-[10]. Its potential applications in quantum information processing have been demonstrated in, such as quantum computation [11], quantum teleportation [12], dense coding [13], quantum cryptographic schemes [14], entanglement swapping [15], remote states preparation [16], and in many pioneering experiments.

To give a proper description and qualify the quantum entanglement for a given quantum state, many entanglement measures have been introduced, such as the entanglement of formation [17] for bipartite quantum systems and concurrence [18] for any multipartite quantum systems. For the two qubit case, the entanglement of formation is proven to be a monotonically increasing function of the concurrence and an elegant formula for the concurrence was derived analytically by Wootters [19]. However, except for bipartite qubit systems and some special symmetric states [20], there have been no explicit analytic formulas of concurrence for arbitrary high-dimensional mixed states, due to the extremizations involved in the computation. Instead of analytic formulas,

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some progress has been made toward the analytical lower bounds of concurrence. A lower bound of concurrence based on local uncertainty relation criterion is derived in [10]. This bound is further optimized in [21]. For arbitrary bipartite quantum states, Refs [22]-[23] provide a detailed proof of an analytical lower bound of concurrence in terms of a different approach that has a close relationship with the distillability of bipartite quantum states. In [23]-[24] the authors presented a lower bound of concurrence by decomposing the joint Hilbert space into many $2 \otimes 2$ and $s \otimes t$ -dimensional subspaces, which improve all the known lower bounds of concurrence.

Based on all lower bounds of bipartite concurrence, nice algorithms and progress has been made towards lower bounds of concurrence for tripartite quantum systems [25]-[26] and other multipartite quantum systems [27] by bipartite partitions of the whole quantum system. One would like to ask naturally if it is possible to improve further the lower bound of concurrence by using tripartite and M -partite concurrences of an N -partite ($M < N$) systems.

In this paper, we first provide lower bounds of concurrence for arbitrary dimensional four-partite systems in terms of tripartite concurrences. Detailed examples are given to show that these bounds are better than the well known existing lower bounds of concurrence. We then generalize lower bound of concurrence to arbitrary multipartite case.

We first recall the definition and some lower bounds of the multipartite concurrence. Let H_i , $i = 1, \dots, N$, be d_i dimensional Hilbert spaces. The concurrence of an N -partite pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \dots \otimes H_N$ is defined by [28],

$$C_N(|\psi\rangle) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_{\alpha} \text{Tr}[\rho_{\alpha}^2]}, \quad (1)$$

where α labels all the different reduced density matrices.

For a mixed multipartite quantum state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in H_1 \otimes H_2 \otimes \dots \otimes H_N$, $p_i \geq 0$, $\sum_i p_i = 1$, the concurrence is given by the convex roof:

$$C_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_N(|\psi_i\rangle), \quad (2)$$

where the minimum is taken over all possible convex decompositions of ρ into pure state ensembles $\{|\psi_i\rangle\}$ with probability distributions $\{p_i\}$.

In [29] the authors obtained lower bounds of multipartite concurrence in terms of the concurrences of bipartite partitioned states of the whole quantum system. For an N -partite quantum pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \dots \otimes H_N$, $\dim H_i = d_i$, $i = 1, \dots, N$, the concurrence of bipartite decomposition between the subsystems $12 \dots M$ and $M+1 \dots N$ is defined by

$$C_2(|\psi\rangle\langle\psi|) = \sqrt{2(1 - \text{Tr}[\rho_{12\dots M}^2])}, \quad (3)$$

where $\rho_{12\dots M} = \text{Tr}_{M+1\dots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over the subsystems $M+1 \dots N$. For a mixed multipartite quantum state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in H_1 \otimes H_2 \otimes \dots \otimes H_N$, the corresponding concurrence $C_2(\rho)$ is given by the convex roof:

$$C_2(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|). \quad (4)$$

A relation between the concurrence (2) and the bipartite concurrence (5) has been presented in [29]: For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes \dots \otimes H_N$ with $N \geq 3$,

the following inequality holds,

$$C_N(\rho) \geq \max 2^{\frac{3-N}{2}} C_2(\rho), \quad (5)$$

where the maximum is taken over all kinds of bipartite concurrences.

In terms of the lower bounds of bipartite concurrence, in [27] further relations between the concurrence (2) and the bipartite concurrence (5) has been obtained:

$$C_N(\rho) \geq \max_{M=1,2,\dots,N-1} \{2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^M - 2} C_2(\rho_M)\} \quad (6)$$

for $N \geq 3$, where the maximum is taken over all kinds of bipartite concurrences for given M . In particular, if $N = 3$, one has $C_3(\rho) \geq \max\{C_2(\rho_1), C_2(\rho_2)\}$. If $N = 4$, one gets $C_4(\rho) \geq \max\{C_2(\rho_1), \frac{\sqrt{3}}{2}C_2(\rho_2), C_2(\rho_3)\}$.

For multi-qubit systems, in [30] the authors get the analytical lower bounds in terms of the monogamy inequality: For any four-qubit mixed quantum state ρ , the concurrence $C(\rho)$ satisfies

$$C^2(\rho) \geq \sum_{i=1}^3 \sum_{j>i}^4 (T_i + T_j) C_{ij}^2(\rho), \quad (7)$$

where

$$\begin{aligned} T_1 &= 1 + \left\{ -\frac{2-x}{2} \middle| \frac{2-x}{2} \right\} + \left\{ -\frac{2-y}{2} \middle| \frac{2-y}{2} \right\} + \left\{ -\frac{2-z}{2} \middle| \frac{2-z}{2} \right\}, \\ T_2 &= 1 + \left\{ \frac{2-x}{2} \middle| -\frac{2-x}{2} \right\} + \left\{ -\frac{y}{2} \middle| \frac{y}{2} \right\} + \left\{ -\frac{z}{2} \middle| \frac{z}{2} \right\}, \\ T_3 &= 1 + \left\{ -\frac{x}{2} \middle| \frac{x}{2} \right\} + \left\{ \frac{2-y}{2} \middle| -\frac{2-y}{2} \right\} + \left\{ \frac{z}{2} \middle| -\frac{2-z}{2} \right\}, \end{aligned}$$

and

$$T_4 = 1 + \left\{ \frac{x}{2} \middle| -\frac{x}{2} \right\} + \left\{ \frac{y}{2} \middle| -\frac{y}{2} \right\} + \left\{ \frac{2-z}{2} \middle| -\frac{2-z}{2} \right\},$$

where the bracket $\{a|b\}$ is so defined such that one may either take the first element a or the second element b from $\{a|b\}$. However, for any given pair a and b , once the first (the second) has been taken, then in a formula one always takes the first (the second) element in all the following brackets containing the same two elements a and b .

In order to improve the lower bounds of concurrence, in the following we consider tripartite concurrence $C_3(\rho)$, instead of the bipartite concurrence $C_2(\rho)$. For an N -partite quantum pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \dots \otimes H_N$, $\dim H_i = d_i$, $i = 1, 2, \dots, N$ ($N \geq 3$), we denote M decomposition among subsystems $\{i^1\}, \{i^2\}, \dots, \{i^{M_1}\}, \{k_1^1, k_2^1\}, \{k_1^2, k_2^2\}, \dots, \{k_1^{M_2}, k_2^{M_2}\}, \dots, \{q_1^1, \dots, q_j^1\}, \dots, \{q_1^{M_j}, \dots, q_j^{M_j}\}$, where $\{i^1, i^2, \dots, i^{M_1}, k_1^1, k_2^1, k_1^2, k_2^2, \dots, k_1^{M_2}, k_2^{M_2}, q_1^1, \dots, q_j^1, \dots, q_1^{M_j}, \dots, q_j^{M_j}\} = \{1, 2, \dots, N\}$ and $\sum_{k=1}^j M_k = M$, $\sum_{k=1}^j k M_k = N$, the concurrence of M -partite decomposition among the above subsystems is given by

$$C_M(|\psi\rangle\langle\psi|) = 2^{1-\frac{M}{2}} \sqrt{(2^M - 2) - \sum_{\alpha} \text{Tr}(\rho_{\alpha}^2)}, \quad (8)$$

where $\alpha = \{\{i^1\}, \{i^2\}, \dots, \{i^{M_1}\}, \{k_1^1, k_2^1\}, \{k_1^2, k_2^2\}, \dots, \{k_1^{M_2}, k_2^{M_2}\}, \dots, \{q_1^1, \dots, q_j^1\}, \dots, \{q_1^{M_j}, \dots, q_j^{M_j}\}\}$.

For example, we can define the concurrence of tripartite decomposition among subsystems $1, 2, \dots, M, M+1, \dots, L$ and $L+1, \dots, N$ as,

$$C_3(|\psi\rangle\langle\psi|) = \sqrt{3 - \text{Tr}(\rho_{12\dots M}^2 + \rho_{M+1\dots L}^2 + \rho_{L+1\dots N}^2)}, \quad (9)$$

where $\rho_{12\dots M} = \text{Tr}_{M+1, \dots, L, L+1, \dots, N}(|\psi\rangle\langle\psi|)$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over the subsystems $M+1, \dots, L, L+1, \dots, N$. Similar definitions apply to $\rho_{M+1\dots L}$ and $\rho_{L+1\dots N}$. The rearrangement of the subsystems are implied naturally, so if take $N=4, M=3$, there are six different dialects of four system: $1|2|34, 1|3|24, 1|4|23, 12|3|4, 13|2|4, 14|2|3$, then we can get the following theorem:

Theorem 1. For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$, then the following inequality holds,

$$C_4^2(\rho) \geq \widetilde{C}_3^2(\rho), \quad (10)$$

where $\widetilde{C}_3^2(\rho) = \frac{1}{6}(C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3}))$.

[Proof]. For a pure multipartite state $|\psi\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$, let $\rho = |\psi\rangle\langle\psi|$, From (1), we have

$$C_4^2(\rho) = \frac{1}{2}(\sum_{i=1}^4(1 - \text{tr}\rho_i^2) + \sum_{i=2}^4(1 - \text{tr}\rho_{1i}^2)). \quad (11)$$

and

$$C_3^2(\rho_{i|j|kl}) = (1 - \text{tr}\rho_i^2) + (1 - \text{tr}\rho_j^2) + (1 - \text{tr}\rho_{kl}^2). \quad (12)$$

where $\rho_i = \text{Tr}_{jkl}(\rho)$, $\rho_j = \text{Tr}_{ikl}(\rho)$, $\rho_{kl} = \text{Tr}_{ij}(\rho)$.

Then from (11) and (12), we have $C_4^2(\rho) \geq \frac{1}{6}(C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3}))$.

Assuming that a mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ attains the minimal decomposition of the multipartite concurrence, one has,

$$\begin{aligned} C_4^2(\rho) &= (\sum_i p_i C_4(|\psi_i\rangle\langle\psi_i|))^2 \\ &\geq (\sum_i p_i \sqrt{\frac{1}{6}(C_3^2((|\psi_i\rangle)_{1|2|34}) + C_3^2((|\psi_i\rangle)_{1|3|24}) + \dots + C_3^2((|\psi_i\rangle)_{14|2|3}))})^2 \\ &\geq (\sum_i p_i \frac{1}{\sqrt{6}} C_3((|\psi_i\rangle)_{1|2|34}))^2 + (\sum_i p_i \frac{1}{\sqrt{6}} C_3((|\psi_i\rangle)_{1|3|24}))^2 + \dots + (\sum_i p_i \frac{1}{\sqrt{6}} C_3((|\psi_i\rangle)_{14|2|3}))^2 \\ &\geq \frac{1}{6}(C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3})), \end{aligned}$$

where the relation $(\sum_j (\sum_i x_{ij})^2)^{\frac{1}{2}} \leq \sum_i (\sum_j x_{ij}^2)^{\frac{1}{2}}$ has been used in second inequality. Therefore, we have (10).

□ It is obvious that our bound is better than the ones given by (5) in [29] and (6) in [27].

We now show some detailed examples. Let us first consider a simple case, the generalized four-qubit GHZ state: $|\psi\rangle = \cos\theta|0000\rangle + \sin\theta|1111\rangle$. We have $C_4(|\psi\rangle) = \sqrt{7\sin^2\theta\cos^2\theta}$. From our lower bound (10), we have $C_4(\rho) \geq \sqrt{6\sin^2\theta\cos^2\theta}$, which is generally greater than the bounds $\sqrt{4\sin^2\theta\cos^2\theta}$ from [27] and $\sqrt{2\sin^2\theta\cos^2\theta}$ from [29].

Now consider the quantum mixed state $\rho = \frac{1-t}{16}I_{16} + t|\phi\rangle\langle\phi|$, with $|\phi\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$, where I_{16} denotes the 16×16 identity matrix. By Theorem 4 in [32], We obtain

$$C^2(\rho_{12|3|4}) \geq \begin{cases} 0, & 0 \leq t \leq \frac{1}{9}, \\ \frac{81t^2-18t+1}{192}, & \frac{1}{9} < t \leq \frac{1}{5}, \\ \frac{181t^2-58t+5}{192}, & \frac{1}{5} < t \leq 1. \end{cases}$$

Also we can get

$$C^2(\rho_{1|3|24}) \geq \begin{cases} 0, & 0 \leq t \leq \frac{1}{5}, \\ \frac{175t^2-70t+7}{192}, & \frac{1}{5} < t \leq 1. \end{cases}$$

Similarly, $C^2(\rho_{1|2|34})$ has the same lower bound as $C^2(\rho_{12|3|4})$, and $C^2(\rho_{1|4|23})$, $C^2(\rho_{13|2|4})$, $C^2(\rho_{14|2|3})$ have the same lower bound as $C^2(\rho_{1|3|24})$. Associated with (10), we have

$$C_4^2(\rho) \geq \begin{cases} 0, & 0 \leq t \leq \frac{1}{9}, \\ \frac{81t^2-18t+1}{576}, & \frac{1}{9} < t \leq \frac{1}{5}, \\ \frac{531t^2-198t+19}{576}, & \frac{1}{5} < t \leq 1. \end{cases}$$

So our result can detect the entanglement of ρ when $\frac{1}{9} < t \leq 1$, see Fig.1. While the lower bound of Theorem 1 in [30] is $C^2(\rho) \geq 0$, when $\frac{1}{9} < t \leq \frac{1}{3}$, which can not detect the entanglement of the above ρ . Also we can found that our lower bound are larger than the lower bound of Theorem 1 in [30] when $\frac{1}{9} < t \leq \frac{111+4\sqrt{106}}{255}$,

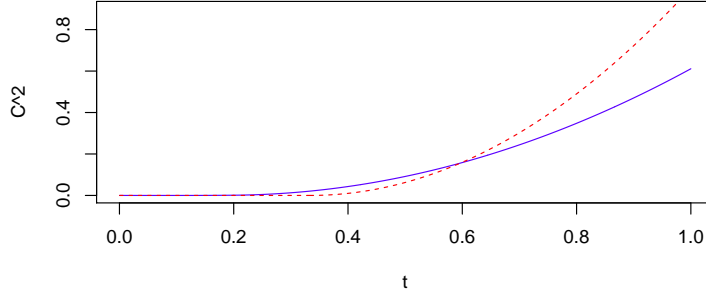


Figure 1: Solid line for the lower bound from (10), which detects the entanglement of ρ when $\frac{1}{9} < t \leq 1$. Dashed line for the lower bound from Theorem 1 in [30]. It detects entanglement only for $t > \frac{1}{3}$.

Similarly, the lower bound of Theorem 1 in [33] is $C^2(\rho) \geq 0$, when $\frac{1}{9} < t \leq \frac{1}{3}$, which can not detect the entanglement of the above ρ . Also we can found that our lower bound are larger than the lower bound of Theorem 1 in [33] when $\frac{1}{9} < t \leq \frac{219+4\sqrt{187}}{579}$.

Remark 1. The definition of concurrence in [30] is different from (1) up to a constant factor $2^{1-N/2}$. In above examples and [33], the difference of the constant factor in defining the concurrence for pure states has already been taken into account.

And if we take $N = 5, M = 3$, there are twenty-five different dialects of five system: $1|2|345, 1|3|245, 1|4|235, 1|5|234, 1|23|45, 1|24|35, 1|25|34, 12|3|45, 12|34|5, 12|4|35, 13|2|45, 13|24|5, 13|4|25, 14|2|35, 14|23|5, 14|3|25, 15|2|34, 15|23|4, 15|3|24, 123|4|5, 134|2|5, 124|3|5, 135|2|4, 125|3|4, 145|2|3$, then we have

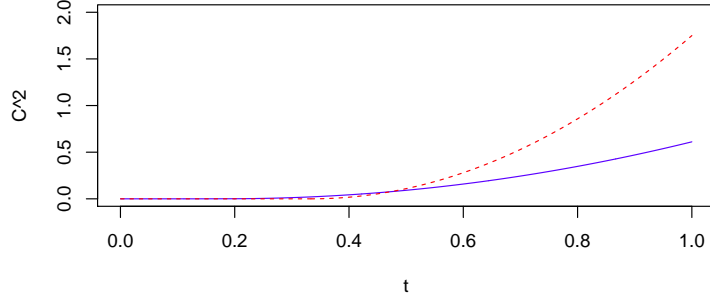


Figure 2: Solid line for the lower bound from (10), which detects the entanglement of ρ when $\frac{1}{9} < t \leq 1$. Dashed line for the lower bound from Theorem 1 in [30]. It detects entanglement only for $t > \frac{1}{3}$.

Theorem 2. For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes H_5$, then the following inequality holds,

$$C_5^2(\rho) \geq \widetilde{C}_3^2(\rho), \quad (13)$$

where $\widetilde{C}_3^2(\rho) = \frac{1}{25}(C_3^2(\rho_{1|2|345}) + C_3^2(\rho_{1|3|245}) + \cdots + C_3^2(\rho_{145|2|3}))$.

[Proof]. For a pure multipartite state $|\psi\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes H_5$, let $\rho = |\psi\rangle\langle\psi|$, From (1), we have

$$C_5^2(\rho) = \frac{1}{4} \left(\sum_{i=1}^5 (1 - \text{tr} \rho_i^2) + \sum_{i=2}^5 (1 - \text{tr} \rho_{1i}^2) + \sum_{i=3}^5 (1 - \text{tr} \rho_{2i}^2) + \sum_{i=4}^5 (1 - \text{tr} \rho_{3i}^2) + (1 - \text{tr} \rho_{45}^2) \right), \quad (14)$$

$$C_3^2(\rho_{i|j|kl}) = (1 - \text{tr} \rho_i^2) + (1 - \text{tr} \rho_{jt}^2) + (1 - \text{tr} \rho_{kl}^2), \quad (15)$$

where $\rho_i = \text{Tr}_{jtkl}(\rho)$, $\rho_{jt} = \text{Tr}_{ikl}(\rho)$, $\rho_{kl} = \text{Tr}_{ijt}(\rho)$, and

$$C_3^2(\rho_{i|j|kls}) = (1 - \text{tr} \rho_i^2) + (1 - \text{tr} \rho_j^2) + (1 - \text{tr} \rho_{kls}^2), \quad (16)$$

where $\rho_i = \text{Tr}_{jkl s}(\rho)$, $\rho_j = \text{Tr}_{ikls}(\rho)$, $\rho_{kls} = \text{Tr}_{ij}(\rho)$.

For a bipartite density matrix $\rho \in H_A \otimes H_B$, from [31], one has

$$1 - \text{Tr}(\rho_{AB}^2) \leq (1 - \text{Tr}(\rho_A^2)) + (1 - \text{Tr}(\rho_B^2)), \quad (17)$$

where $\rho_A = \text{Tr}_B(\rho_{AB})$, $\rho_B = \text{Tr}_A(\rho_{AB})$.

Then from (14), (15), (16) and (17), we have $C_5^2(\rho) \geq \frac{1}{25}(C_3^2(\rho_{1|2|345}) + C_3^2(\rho_{1|3|245}) + \cdots + C_3^2(\rho_{145|2|3}))$.

Assuming that a mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ attains the minimal decomposition of the multipartite concurrence, one has,

$$\begin{aligned} C_5^2(\rho) &= (\sum_i p_i C_5(|\psi_i\rangle\langle\psi_i|))^2 \\ &\geq \left(\sum_i p_i \sqrt{\frac{1}{25}(C_3^2((|\psi_i\rangle)_{1|2|345}) + C_3^2((|\psi_i\rangle)_{1|3|245}) + \cdots + C_3^2((|\psi_i\rangle)_{145|2|3}))} \right)^2 \\ &\geq \left(\sum_i p_i \frac{1}{5} C_3((|\psi_i\rangle)_{1|2|345}) \right)^2 + \left(\sum_i p_i \frac{1}{5} C_3((|\psi_i\rangle)_{1|3|245}) \right)^2 + \cdots + \left(\sum_i p_i \frac{1}{5} C_3((|\psi_i\rangle)_{145|2|3}) \right)^2 \\ &\geq \frac{1}{25} (C_3^2(\rho_{1|2|345}) + C_3^2(\rho_{1|3|245}) + \cdots + C_3^2(\rho_{145|2|3})), \end{aligned}$$

where the relation $(\sum_j (\sum_i x_{ij})^2)^{\frac{1}{2}} \leq \sum_i (\sum_j x_{ij}^2)^{\frac{1}{2}}$ has been used in second inequality. Therefore, we have (13).

If we take $N = 5, M = 4$, there are ten different dialects of five system: $1|2|3|4|5, 1|2|4|3|5, 1|2|5|3|4, 1|2|3|4|5, 1|2|4|3|5, 1|2|5|3|4, 1|3|2|4|5, 1|4|2|3|5, 1|5|2|3|4$, similar to Theorem2, we can get

Theorem 3. *For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes H_5$, then the following inequality holds,*

$$C_5^2(\rho) \geq \widetilde{C}_4^2(\rho), \quad (18)$$

where $\widetilde{C}_4^2(\rho) = \frac{1}{10}(C_4^2(\rho_{1|2|3|4|5}) + C_4^2(\rho_{1|2|4|3|5}) + \cdots + C_4^2(\rho_{15|2|3|4}))$.

Now we generalize our results to N -partite systems ($N > 4$). For a given N -partite state, $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$, we can define the M -partite concurrences $C_M(|\psi\rangle\langle\psi|)$ associated with the corresponding decompositions among subsystems. Similar to the result (10) for tripartite decomposition and the result (18) for four-partite decomposition, We have $C_N^2(\rho) \geq \widetilde{C}_M^2(\rho)$, where $\widetilde{C}_M^2(\rho)$ takes average over all possible square M -partite concurrences. Generally, we obtain

Theorem 4.

$$C_N^2(\rho) \geq s_1 \max\{\widetilde{C}_{N-1}^2(\rho)\} + s_2 \max\{\widetilde{C}_{N-2}^2(\rho)\} + \cdots + s_{N-2} \max\{\widetilde{C}_2^2(\rho)\},$$

where $\sum_{i=1}^{N-2} s_i = 1, s_i \geq 0$.

In summary, we have presented an approach to derive lower bounds of concurrence for arbitrary dimensional N -partite systems based on sub M -partite ($M = 3, \dots, N-1$) concurrences. Lower bounds of concurrence for four-partite mixed states have been studied in detail in terms of the tripartite concurrences. By detailed examples we have shown that this bound is better than other existing lower bounds of concurrence.

Above all, in [23]-[25] lower bounds of concurrence for high dimensional systems have been presented based on the concurrences of sub-dimensional states, by decomposing the joint Hilbert space into lower dimensional subspaces. For high dimensional multipartite systems, it would be useful to use the concurrences of both sub-dimensional states and sub-partite states. An optimal lower bound could be obtained by repeatedly using the concurrences of sub-dimensional and sub-partite states in an suitable order.

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References

- [1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information(Cambridge University Press, Cambridge, 2000).
- [2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777(1935).
- [3] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature(London)416, 608(2002).
- [4] R. F. Werner, Phys. Rev. A 40, 4277(1989).
- [5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865(2009).
- [6] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. 92, 167902(2004);
F. Mintert, Ph.D. thesis, Measures and dynamics of entangled states, Munich University, Munich, 2004.
- [7] K. Chen, S. Alberverio, and S. M. Fei, Phys. Rev. Lett. 95, 040504(2005).
- [8] H. P. Breuer, J. Phys. A: Math. Gen. 39, 11847 (2006).

- [9] H. P. Breuer, Phys. Rev. Lett. 97, 080501(2006).
- [10] J. I. de Vicente, Phys. Rev. A 75, 052320(2007).
- [11] See, for example, D. P. DiVincenzo, Science 270, 255(1995).
- [12] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993);
S. Albeverio, S. M. Fei and Y. L. Yang, Phys. Rev. A 66, 012301(2002).
- [13] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881(1992).
- [14] A. Ekert, Phys. Rev. Lett. 67, 661(1991);
C. A. Fuchs, N. Gisin, R. B. Griffiths, C. S. Niu, and A. Peres, Phys. Rev. A 56, 1163(1997).
- [15] M. Żukowski, A. Zeilinger, M. A. Horne and A. K. Ekert, Phys. Rev. Lett. 71, 4287(1993);
S. Bose, V. Vedral and P. L. Knight, Phys. Rev. A 57, 822(1998); 60, 194(1999);
B. S. Shi, Y. K. Jiang, G. C. Guo, Phys. Rev. A 62, 054301(2000).
- [16] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J.A. Smolin, B. M. Terhal and W. K. Wootters, Phys. Rev. Lett. 87, 077902(2001);
D. W. Leung and P. W. Shor, Phys. Rev. Lett. 90, 127905(2003).
- [17] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54,3824(1996);
M. B. Plenio and S. Virmani, Quant. Inf. Comput. 7, 1(2007).
- [18] A. Uhlmann, Phys. Rev. A 62, 032307(2000);
P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A 64, 042315(2001);
S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclassical Opt. 3, 223(2001).
- [19] W. K. Wootters, Phys. Rev. Lett. 80, 2245(1998).
- [20] B. M. Terhal and K. G. H. Vollbrecht, Phys. Rev. Lett. 85, 2625(2000);
S. M. Fei, J. Jost, X. Q. Li-Jost, and G. F. Wang, Phys. Lett. A 310, 333(2003);
P. Rungta and C. M. Caves, Phys. Rev. A 67, 012307(2003);
S. M. Fei and X. Q. Li-Jost, Rep. Math. Phys. 53, 195(2004);
S. M. Fei, Z. X. Wang, and H. Zhao, Phys. Lett. A 329, 414(2004).
- [21] C. J. Zhang, Y. S. Zhang, S. Zhang, and G. C. Guo, Phys. Rev. A 76, 012334(2007).
- [22] E. Gerjuoy, Phys. Rev. A 67,052308(2003).
- [23] Y. C. Ou, H. Fan, and S. M. Fei, Phys. Rev. A 78, 012311(2008).
- [24] M. J. Zhao, X. N. Zhu, S. M. Fei, and X. Q. Li-Jost, Phys. Rev. A 84, 062322(2011).
- [25] X. N. Zhu, M. J. Zhao and S. M. Fei, Phys. Rev. A 86, 022307(2012).
- [26] X. H. Gao, S. M. Fei, and K. Wu, Phys. Rev. A 74, 050303(2006).
- [27] X. N. Zhu, M. Li, and S. M. Fei, Aip Conf. Proc. Advances in Quan. Theory, 1424(2012).
- [28] L. Aolita and F. Mintert, Phys. Rev. Lett. 97, 050501(2006);
A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 93, 230501(2004).
- [29] M. Li, S. M. Fei and Z. X. Wang, Rep. Math. Phys. 65, 289-296(2010).
- [30] X. N. Zhu, S. M. Fei, Quantum Inform. Processing 13, 815-823(2014).
- [31] J. M. Cai, Z. W. Zhou, S. Zhang and G. C. Guo, Phys. Rev. A 75, 052324(2007).
- [32] W. Chen, S. M. Fei, Z. J. Zheng, Quantum Inform. Processing (2016), Doi:10.1007/s11128-016-1369-x.
- [33] X. F. Qi, T. Gao, F. l. Yan, arXiv:1605.05000.